1 Quiz 9 (Apr 8) solutions

• 1. For the Leslie matrix $\begin{bmatrix} 2 & 4 \\ 0.1 & 0.2 \end{bmatrix}$, find the strictly dominant eigenvalue and the normalized eigenvector corresponding to it.

Well, $\det(\begin{bmatrix} 2-\lambda & 4 \\ 0.1 & 0.2-\lambda \end{bmatrix}) = (2-\lambda)(0.2-\lambda) - 0.1 * 4 = (0.4 - (2+0.2)\lambda + \lambda^2) - 0.4 = \lambda^2 - 2.2\lambda = \lambda(\lambda - 2.2) = 0$ tells us that $\lambda_1 = 0$, $\lambda_2 = 2.2$ are our two eigenvalues, and since $|\lambda_1| = 0 < |\lambda_2| = 2.2$, $\lambda_2 = 2.2$ is our strictly dominant eigenvalue.

To find the normalized eigenvector corresponding to $\lambda_2 = 2.2$, we set up the equation $(A - \lambda_2 I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We have $A - \lambda_2 I = \begin{bmatrix} 2-2.2 & 4 \\ 0.1 & 0.2-2.2 \end{bmatrix} = \begin{bmatrix} 0.2 & 4 \\ 0.1 & -2 \end{bmatrix}$ so the equation is $\begin{bmatrix} 0.2 & 4 \\ 0.1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.2x + 4y \\ 0.1x - 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Using the equation $0.1x - 2y = 0$, we have that $y = \frac{0.1}{2}x = 0.05x$, so $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0.05x \end{bmatrix}$ is a 2.2-eigenvector for any x. Choosing $x = 1$, $\begin{bmatrix} 1 \\ 0.05 \end{bmatrix}$ is an eigenvector, and to normalize we add and divide: so $\frac{1}{1+0.05}\begin{bmatrix} 1 \\ 0.05 \end{bmatrix} = \begin{bmatrix} \frac{1}{1.05} \\ \frac{0.05}{1.05} \end{bmatrix} \approx \begin{bmatrix} 0.95 \\ 0.05 \end{bmatrix}$ is our normalized eigenvector corresponding to $\lambda_2 = 2.2$.

• 2. A population of insects is divided into two stage categories. The females in the first category or stage produce on average 2 female offspring and have a survival rate of 25% to stage 2. The females in stage 2 produce on average 5 female offspring and die after one time period in stage 2. What is the Leslie matrix?

If $S_1(t)$, $S_2(t)$ is the population at time t of stage 1 and stage 2 respectively, we assume that offspring is born into stage 1. Then we have that that $S_1(t+1) = 2S_1(t) + 5S_2(t)$ and $S_2(t+1) = 0.25S_1(t)$. Then our Leslie matrix would be

$$\begin{array}{c c} & \begin{array}{cc} S_1(t) & S_2(t) \end{array} \\ \begin{array}{c} S_1(t+1) \\ S_2(t+1) \end{array} & \begin{pmatrix} 2 & 5 \\ 0.25 & 0 \end{pmatrix} \end{array}$$

2 Worksheet (Apr 8) solutions

Find the strictly dominant eigenvalue and the normalized eigenvector corresponding to it. Since $x(k) = A^k x(0)$ is characterized by the dominant term $r^k cv$, do we have exponential growth, decay, or convergence to equilibrium?

• (a) $\begin{bmatrix} 0.5 & 0.9 \\ 0.4 & 0 \end{bmatrix}$

First, to find the eigenvalues, we solve $\det(A - \lambda I) = 0$: $\det(\begin{bmatrix} 0.5-\lambda & 0.9 \\ 0.4 & 0-\lambda \end{bmatrix}) = (0.5-\lambda)(0-\lambda) - 0.9 * 0.4 = \lambda^2 - 0.5\lambda - 0.36 = (\lambda - 0.9)(\lambda + 0.4) = 0$. Thus $\lambda_1 = 0.9$ is the strictly dominant eigenvalue.
To find the normalized eigenvector corresponding to it, we solve the equation $(A - \lambda_1 I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since $(A - \lambda_1 I) = \begin{bmatrix} 0.5-0.9 & 0.9 \\ 0.4 & 0-0.9 \end{bmatrix} = \begin{bmatrix} -0.4 & 0.9 \\ 0.4 & -0.9 \end{bmatrix}$, $\begin{bmatrix} -0.4 & 0.9 \\ 0.4 & -0.9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.4x + 0.9y \\ 0.4x - 0.9y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So we use the equation $0.4x - 0.9y = 0$ to see that $\frac{4}{9}x = y$. Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{4}{9}x \end{bmatrix}$ is an eigenvector for any x, so choose $x = 9$ for computations. Then $\begin{bmatrix} 9 \\ 4 \end{bmatrix}$ is an eigenvector with sum $9 + 4 = 13$, so our normalized eigenvector is $\begin{bmatrix} \frac{9}{13} \\ \frac{4}{13} \end{bmatrix} \approx \begin{bmatrix} 0.69 \\ 0.31 \end{bmatrix}$.

- (b) $\begin{bmatrix} 0.7 & 0.2 \\ 0.2 & 0.4 \end{bmatrix}$

First, to find the eigenvalues, we solve $\det(A - \lambda I) = 0$: $\det(\begin{bmatrix} 0.7-\lambda & 0.2 \\ 0.2 & 0.4-\lambda \end{bmatrix}) = (0.7-\lambda)(0.4-\lambda) - 0.2 * 0.2 = (\lambda^2 - (0.7+0.4)\lambda + 0.28) - 0.04 = \lambda^2 - 0.11\lambda + 0.24 = (\lambda - 0.8)(\lambda - 0.3) = 0$. Thus $\lambda_1 = 0.8$ is the strictly dominant eigenvalue.
To find the normalized eigenvector corresponding to it, we solve the equation $(A - \lambda_1 I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since $(A - \lambda_1 I) = \begin{bmatrix} 0.7-0.8 & 0.2 \\ 0.2 & 0.4-0.8 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.2 \\ 0.2 & -0.4 \end{bmatrix}$, $\begin{bmatrix} -0.1 & 0.2 \\ 0.2 & -0.4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.1x + 0.2y \\ 0.2x - 0.4y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So we use the equation $-0.1x + 0.2y = 0$ to see that $\frac{1}{2}x = y$. Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{1}{2}x \end{bmatrix}$ is an eigenvector for any x, so choose $x = 2$ for computations. Then $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector with sum $2 + 1 = 3$, so our normalized eigenvector is $\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \approx \begin{bmatrix} 0.67 \\ 0.33 \end{bmatrix}$.

- (c) $\begin{bmatrix} 0.1 & 1.2 \\ 0.4 & 0.3 \end{bmatrix}$

First, to find the eigenvalues, we solve $\det(A - \lambda I) = 0$: $\det(\begin{bmatrix} 0.1-\lambda & 1.2 \\ 0.4 & 0.3-\lambda \end{bmatrix}) = (0.1-\lambda)(0.3-\lambda) - 0.4 * 1.2 = (\lambda^2 - (0.1+0.3)\lambda + .03) - 0.48 = \lambda^2 - 0.4\lambda - 0.45 = (\lambda - 0.9)(\lambda + 0.5) = 0$. Thus $\lambda_1 = 0.9$ is the strictly dominant eigenvalue.
To find the normalized eigenvector corresponding to it, we solve the equation $(A - \lambda_1 I)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Since $(A - \lambda_1 I) = \begin{bmatrix} 0.1-0.9 & 1.2 \\ 0.4 & 0.3-0.9 \end{bmatrix} = \begin{bmatrix} -0.8 & 1.2 \\ 0.4 & -0.6 \end{bmatrix}$, $\begin{bmatrix} -0.8 & 1.2 \\ 0.4 & -0.6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.8x + 1.2y \\ 0.4x - 0.6y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So we use the equation $0.4x - 0.6y = 0$ to see that $\frac{2}{3}x = y$. Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ \frac{2}{3}x \end{bmatrix}$ is an eigenvector for any x, so choose $x = 3$ for computations. Then $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ is an eigenvector with sum $3 + 2 = 5$, so our normalized eigenvector is $\begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$.